The Measure of the Metric

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Every distance on $\mathbb{R}^D$ requires the Euclidean metric δ_{ij}.
Every physical theory on $\mathbb{R}^D$ inherits it.
What does δ_{ij} determine?

The Euclidean metric δ_{ij} is inside every dot product $\delta_{ij}v^i w^j$, every norm $|\mathbf{x}|^2 = \delta_{ij}x^i x^j$, every kinetic energy $\frac{1}{2}m\,\delta_{ij}v^i v^j$. It defines what "length" means. The isometry group preserving δ_{ij} is the Euclidean group $\mathrm{ISO}(D)$: rotations and translations. This is Euclidean geometry (Euclid, Riemann 1854).

Euclidean geometry is the symmetry of this tensor. We now seek the measure it determines. The resulting measure is unique. Every calculation below uses mathematical tools available before 1910.

The operator

The metric δ_{ij} is symmetric and rank 2. It determines a unique second-order differential operator: the Laplacian $\Delta = \delta^{ij}\partial_i\partial_j$. No first-order $\mathrm{SO}(D)$-invariant operator on scalar functions exists: it would require an invariant vector, but $\mathrm{SO}(D)$ has none. Translation invariance forces constant coefficients. The sign is forced: $S = \frac{1}{2}\int(|\nabla\phi|^2 + m^2\phi^2)\,d^D x \geq 0$ after integration by parts. The wrong sign gives S unbounded below. The most general $\mathrm{ISO}(D)$-invariant second-order operator on scalars with non-negative action is

$$-\Delta + m^2, \qquad m^2 \geq 0.$$

Higher powers $(-\Delta+m^2)^n$ with $n \geq 2$ are excluded. The double-pole residue of $(-\Delta+m^2)^{-2}$ gives two mode functions $e^{-\omega s}$ and $se^{-\omega s}$; their inner-product kernel at distinct points $a \neq b$ has Gram determinant $-\omega^2(a-b)^2 e^{-2\omega(a+b)} < 0$. The reflected kernel is indefinite; no positive Hilbert-space inner product exists, and the measure fails reflection positivity. Without it, the reconstructed inner product admits $\|P_\alpha\psi\|^2 < 0$: negative probabilities. Products of distinct simple poles decompose with opposite-sign residues: indefinite again. Every $n \geq 2$ fails. Fractional powers $(-\Delta+m^2)^\alpha$, $0 < \alpha < 1$, have positive kernels but are non-local; δ_{ij} is defined pointwise and determines only differential operators.

One operator survives. On flat $\mathbb{R}^D$ the spin connection is trivial; the Laplacian on any vector bundle acts as the scalar Laplacian on each component. The argument holds for every spin.

The measure

The action $S[\phi] = \frac{1}{2}\int \phi\,(-\Delta + m^2)\phi\; d^D x$ on $\phi\colon \mathbb{R}^D \to \mathbb{R}$ is quadratic and non-negative. The Gaussian measure $e^{-S/\hbar}\mathcal{D}\phi$, with $\hbar > 0$, has two-point function

$$K(x, y) = \hbar\,(-\Delta + m^2)^{-1}(x, y).$$

This measure is consistent. The operator $-\Delta+m^2$ is ISO(D)-invariant by construction. The kernel $K(x, y) = K(y, x)$ is symmetric, inherited from self-adjointness. For $m > 0$, K decays as $e^{-m|x-y|}$: distant regions decouple exponentially.

Single out one coordinate x_0 as Euclidean time. The reflected kernel, evaluated at positive times $s_i, s_j > 0$, factorises:

$$K(s_i + s_j) = \frac{\hbar\, e^{-\omega(s_i+s_j)}}{2\omega} = v_i\, v_j, \qquad v_k = \sqrt{\frac{\hbar}{2\omega}}\; e^{-\omega s_k}.$$

A product of positive functions is positive-semidefinite. This is the condition that allows analytic continuation to real time. It holds for the unique operator $-\Delta + m^2$ and fails for every higher-order alternative, by the same Gram determinant that excludes higher-order field equations in the Lorentzian case. The positive kernel that selects the field equations simultaneously defines the inner product of the Hilbert space: one condition, two readings.

The continuation

The positive kernel determines the analytic continuation to real time. The reflected kernel defines a contraction semigroup: $K(s + \tau) = e^{-\omega\tau} K(s)$ for $\tau > 0$. The function $e^{-\omega\tau}$ is entire in τ. Its restriction to $\tau = it$ gives

$$e^{-\omega\tau} \;\longrightarrow\; e^{-i\omega t}.$$

Contraction becomes oscillation. This requires a square root of -1 in the scalar field. The four normed division algebras over $\mathbb{R}$ (Frobenius 1878, Hurwitz 1898) are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

Over $\mathbb{R}$: the continuation requires a complex structure $J^2 = -1$; that is extra data not determined by the metric.

Over the quaternions $\mathbb{H}$: three square roots of -1 give three inequivalent unitary groups; the choice is undetermined. The tensor product of two quaternionic spaces is not quaternionic; composition of independent systems requires additional choices not determined by the metric.

Over the octonions $\mathbb{O}$: non-associative; no operator algebra.

Over $\mathbb{C}$: one square root of -1 up to conjugation. Unique. The non-trivial element $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts by $z \mapsto \bar{z}$.

A bilinear inner product over $\mathbb{C}$ gives $\langle if, if\rangle = i^2\langle f, f\rangle = -\langle f, f\rangle$: indefinite, just as the wrong-sign Gram determinant was indefinite. Sesquilinearity is the unique positive-definite alternative: $\langle if, if\rangle = \bar{i}\cdot i\,\langle f, f\rangle = +\langle f, f\rangle$. The inner product must use σ.

The Galois automorphism determines everything that follows.

The contraction semigroup $e^{-\omega\tau}$ becomes the unitary group $e^{-i\omega t}$, acting on a complex Hilbert space with sesquilinear inner product. Its self-adjoint generator $H = \omega \geq 0$ has non-negative spectrum: energy is bounded below.

The Euclidean group continues to the Poincaré group: $\mathrm{ISO}(D) \to \mathrm{ISO}(D{-}1,1)$. The Euclidean kernel satisfies $K(x,y) = K(y,x)$; under continuation, this gives $[\phi(x), \phi(y)] = 0$ for spacelike-separated points.

The Gaussian factorisation of the measure gives Fock space: multiparticle states and entanglement. The action $S \geq 0$ has its minimum at $\phi = 0$: the vacuum is unique.

The propagator pole $k^2 + m^2 = 0$, continued to real time, gives $E^2 = |\mathbf{p}|^2 c^2 + m^2 c^4$. In the rest frame,

$$E = mc^2.$$

The Hilbert space was built from the measure. For any projection P_α, the probability the measure assigns is $\|P_\alpha \psi\|^2$: the Born rule. It is not imposed on the Hilbert space. It is what the Hilbert space was built from. The measure is a probability distribution over definite field configurations; measurement is conditioning on observed data.

The translation generators in the position representation satisfy $[\hat{x}^i, \hat{p}_j] = i\hbar\, \delta^i{}_j$. The Cauchy–Schwarz inequality gives

$$\Delta x\, \Delta p \;\geq\; \frac{\hbar}{2}.$$

The kernel $K = \hbar\, e^{-\omega s}/(2\omega)$ is strictly positive for all $s > 0$. Correlations decay through any barrier but never reach zero. Tunnelling is the positivity of the kernel.

In $D \geq 3$, analytic continuation in $\mathrm{SO}(D, \mathbb{C})$ connects the sign under 2π rotation to the sign under particle exchange: $(-1)^{2s}$. Bosons commute; fermions anticommute. The wrong statistics give energy unbounded below: the same positivity that selected the operator selects the statistics. The Clifford algebra $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ and its spinor representations are determined by δ_{ij}; the Dirac operator satisfies $(\gamma^i \partial_i)^2 = \Delta$, reducing to the same unique operator.

The semigroup $e^{-\omega\tau}$ is the Boltzmann factor $e^{-\beta H}$ under $\tau = \beta$: Euclidean time is inverse temperature. The same measure, under σ, gives the Feynman integral $e^{iS/\hbar}$. Three frameworks read one object:

$e^{-S/\hbar}$ with $\tau = \beta$: the measure as thermal weight (Boltzmann).

$e^{-S/\hbar}$ under σ: the measure continued to real time (Feynman).

$\|P_\alpha \psi\|^2$ from the inner product K: the measure as probability (Born).

The action of σ on Lorentz representations maps $(\frac{1}{2}, 0) \leftrightarrow (0, \frac{1}{2})^*$: CPT.

The result

The Euclidean metric on $\mathbb{R}^D$ determines a unique second-order operator and a unique reflection-positive Gaussian measure. Its analytic continuation to real time determines the kinematic framework of quantum mechanics: complex Hilbert space, sesquilinear inner product, unitary evolution, and the Born rule. The evolution equation is

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = H\psi}$$

with two free parameters $m \geq 0$ and $\hbar > 0$.

This is quantum mechanics. $\square$